DESALINATION OF A WET SOIL

V. I. Pen'kovskii

UDC 624.131.6:532.72

A highly water-retentive soil has most of the dissolved salts in the pore solution in the aeration zone, and this solution is bound to the soil skeleton. The water becomes immobile in such a soil below some minimum water content [1], which is called the minimum field capacity or the capillary-rupture water content, at which level the space occupied by the liquid phase becomes discontinuous. Under these conditions, the transfer of salt between layers by convection and diffusion becomes negligible, and so the salt-distribution curves may persist unchanged for long periods, as is frequently observed.

When desalination is started, the initial effect is that continuity is restored in the liquid zone, and thus the salts become redistributed in response to convection and diffusion.

Let m_2 be the threshold water content as defined above, with m_1 the amount of water capable of moving under gravity. If the soil is desalinated by irrigation at a rate $v < k$, where k is the soil infiltration coefficient, one can estimate m_1, for instance, by means of a formula for the flow rate in an incompletely saturated pore system [2]:

$$m_1 = (m - m_2)(v/k)^{1/n},$$

where m is the porosity and $n \approx 3.5$ is an empirical constant. If the desalination involves flooding the surface, then $v \geq k$, while $m_1 = m - m_2$ on account of the physical meaning of the saturation coefficient (or water-release coefficient).

We consider here the redistribution of the salt in the desalination zone during the initial stage (from the moment when the water is supplied and a moving wetted front $x = x_0(t)$ appears), and also in the final stage (the water input ceases, and a free surface $x = x_1(t)$ appears within the soil, at which level there is a specific salt content). In the latter case, the final salt content of the soil at a given point is determined by the concentration of the pore solution retained by the skeleton.

1. Initial Stage. Let the x axis have its origin at the surface and be directed into the soil.

Fick's law gives the salt flux through unit area perpendicular to the x axis as

$$q = -Dc_x + vc,$$

where D is the salt diffusion coefficient for a medium with a given pore saturation, c is concentration, and v is filtration rate, while we take the law of conservation of the salt for each elementary volume as

$$-q_x = [(m_1 + m_2)c]_t$$

which implies that $0 < x < x_0(t) = \int_0^t \frac{v}{m_1} dt$ in the region where the solution is moving, and there the function c(x, t) should satisfy the differential equation

$$(Dc_x - vc)_x = [(m_1 + m_2)c]_t .$$

The salt mass flux q_0 through an area moving in the wetted zone $0 < x < x_0(t)$ along the x axis with a speed v_n is given by

---

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 137-141, January-February, 1977. Original article submitted March 22, 1976.

$$q_0 = q - v_n(m_1 + m_2)c.$$

In particular (for $v_n = dx_0/dt = v/m_1$), we get

$$q_0 = - Dc_x - m_2vc/m_1.$$

Let N(x) be the initial concentration of the pore solution bound to the soil; then through an area in the region $x_0 < x < \infty$ there moves a salt flux $v_n m_2 N(x)$ at a speed v_n; we put $v_n = v/m_1$ and equate this flux to q_0 to get the boundary condition at the mobile front for $x = x_0(t)$:

$$Dc_x + m_2vc/m_1 = m_2vN(x_0(t))/m_1.$$

In what follows we assume that m_1, m_2, v and D are constants; we introduce the new independent variables

$$\xi = vx/D,\ \tau = v^2t/(m_1D),$$

and for desalination by flooding with fresh water we have

$$\begin{cases} c_{\xi\xi} - c_\xi = \lambda^{-1}c_\tau,\ 0<\xi<\tau; \\ c = 0 \quad \text{for} \quad \xi = 0,\ \tau > 0; \\ c_\xi + m_2c/m_1 = \varphi(\tau) \quad \text{for} \quad \xi = \tau > 0, \end{cases} \tag{1.1}$$

where

$$\lambda = m_1/(m_1 + m_2);\ \varphi(\tau) = m_2N(D\tau/v)m_1.$$

We represent the target function in the form

$$c(\xi, \tau) = u(\xi, \tau) \exp(\xi/2 - \lambda\tau/4),$$

and then (1.1) reduces to

$$\begin{cases} u_{\xi\xi} = \lambda^{-1}u_\tau,\ 0<\xi<\tau,\ \tau>0; \\ u = 0 \quad \text{for} \quad \xi = 0,\ \tau > 0; \\ u_\xi + \delta u = \varphi(\tau)\exp[\tau(\lambda - 2)/4] \quad \text{for} \quad \xi = \tau > 0 \end{cases} \tag{1.2}$$

$$(\delta = 1/2 + m_2/m_1).$$

The function $u(\xi, \tau)$ is sought as the solution to the initial boundary-value problem for the thermal-conduction equation in the region $\xi > 0$, $\tau > 0$:

$$u(\xi, \tau) = \frac{1}{2\sqrt{\pi\lambda\tau}} \int_0^\infty \rho(s) \left\{ \exp\left[-\frac{(\xi - s)^2}{4\lambda\tau}\right] - \exp\left[-\frac{(\xi + s)^2}{4\lambda\tau}\right] \right\} ds, \tag{1.3}$$

where $\rho(s)$ is to be determined.

We satisfy the condition at $\xi = \tau$, to get $\rho(s)$ as defined by the integrodifferential equation

$$\frac{1}{\sqrt{\pi\lambda\tau}} \int_0^\infty F(s, \lambda) \exp\left(-\frac{s^2}{4\lambda\tau}\right) ds = \varphi(\tau) e^{\gamma\tau}$$
$$\left(F(s, \lambda) = \frac{d\rho}{ds} \operatorname{ch} \frac{s}{2\lambda} + \delta\rho \operatorname{sh} \frac{s}{2\lambda};\ \gamma = \frac{(1-\lambda)^2}{4\lambda}\right) \tag{1.4}$$

subject to the initial condition

$$\rho(0) = 0, \tag{1.5}$$

which follows from consistency between the limits at $\xi = \tau = 0$; by

$$L_p(\varphi) = \int_0^\infty \varphi(t)\, e^{-pt} dt = \Phi(p),$$

$$L_s^{-1}(\Phi) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Phi(p)\, e^{ps} dp = \varphi(s),$$

we denote the forward and reverse Laplace transforms, respectively. Then we insert the integral operator in (1.4) to get [3]

$$F(s, \lambda) = L_s^{-1} [\lambda p L_{\lambda p^2 - \gamma}(\varphi)].$$

We integrate this equation subject to (1.5) to get

$$\rho(s) = \left(\operatorname{ch} \frac{s}{2\lambda}\right)^{\lambda-2} \int_0^s F(z, \lambda) \left(\operatorname{ch} \frac{z}{2\lambda}\right)^{1-\lambda} dz. \tag{1.6}$$

Then substitution of (1.6) into (1.3) gives $u(\xi, \tau)$, satisfying all the conditions of (1.2).

2. Salt Redistribution during Gravitational Water Flow. Let $x = x_1(t) = v(t - t_0)/m_1$ be the position of the descending groundwater level, with the moment of descent below the free surface taken as origin ($t_0 = 0$).

Consider an area moving along the x axis in the $0 < x < x_1(t)$ zone with speed v/m_1, for which the salt mass flux is $vm_2c_1(x)/m_1$, where $c_1(x)$ is the salt distribution in the pore space when the water has flowed away under gravitation. We equate this to the flux q_0 in the zone $x_1 < x < \infty$ and bear in mind that $c = c_1(x)$ for $x = x_1(t)$ to get the following condition at the mobile boundary:

$$Dc_x + m_2vc/m_1 = m_2vc/m_1.$$

Then this stage is simulated by

$$\begin{cases} c_{\xi\xi} - c_\xi = \lambda^{-1} c_\tau, \ 0 < \tau < \xi < \infty, \\ c_\xi = 0 \quad \text{for} \ \xi = \tau > 0 \\ c = c_0(\xi) \quad \text{for} \ \xi > 0, \ \tau = 0, \end{cases} \tag{2.1}$$

where $c_0(\xi)$ is the initial concentration distribution, i.e., the one produced at the end of the first desalination stage. We assume that $c_0(\xi)$ is differentiable (this restriction is unimportant and can be weakened). In terms of the function $u(\xi, \tau) = c(\xi, \tau) \exp(-\xi/2 + \lambda\tau/4)$, the problem of (2.1) takes the form

$$\begin{gathered} u_{\xi\xi} = \lambda^{-1} u_\tau, \quad 0 < \tau < \xi < \infty; \\ u_\xi + u/2 = 0 \quad \text{for} \ \xi = \tau > 0; \\ u = f(\xi) = c_0(\xi) \exp(-\xi/2) \ \text{for} \ \xi > 0, \ \tau = 0. \end{gathered} \tag{2.2}$$

The solution is represented as a Poisson integral:

$$u(\xi, \tau) = \frac{1}{2\sqrt{\pi\lambda\tau}} \left\{ \int_{-\infty}^0 \rho_0(s) \exp\left[-\frac{(\xi - s)^2}{4\lambda\tau}\right] + \int_0^\infty f(s) \exp\left[-\frac{(\xi - s)^2}{4\lambda\tau}\right] ds \right\}, \tag{2.3}$$

where $\rho_0(s)$ is some continuous extension of $f(s)$ for $s < 0$, where

$$\rho_0(0) = f(0). \tag{2.4}$$

Then the function of (2.3) satisfies the equation and the condition for $\tau = 0$ for the problem of (2.2); we put $\rho(s) = \rho_0(-s)$ and rewrite (2.3) as

$$u(\xi, \tau) = \frac{1}{2\sqrt{\pi\lambda\tau}} \int_0^\infty \left\{ \rho(s) \exp\left[-\frac{(\xi + s)^2}{4\lambda\tau}\right] + f(s) \exp\left[-\frac{(\xi - s)^2}{4\lambda\tau}\right] \right\} ds.$$

We integrate by parts with (2.4) and note that

$$\frac{\partial}{\partial \xi}\exp\left[-\frac{(\xi\pm s)^2}{4\lambda\tau}\right]=\pm\frac{\partial}{\partial s}\exp\left[-\frac{(\xi\pm s)^2}{4\lambda\tau}\right],$$

to put the condition at $\xi = \tau$ for $u(\xi, \tau)$ in (2.2) as

$$[u_\xi+u/2]_{\xi=\tau}=\frac{\exp(-\tau/4\lambda)}{2\sqrt{\pi\lambda\tau}}\int_0^\infty\left[\left(\frac{df}{ds}+\frac{1}{2}f\right)e^{\frac{s}{2\lambda}}-\left(\frac{d\rho}{ds}-\frac{1}{2}\rho\right)e^{-\frac{s}{2\lambda}}\right]e^{-\frac{s^2}{4\lambda\tau}}ds=0.$$

This must be obeyed for any $\tau > 0$, and for this purpose it is sufficient that $\rho(s)$ satisfies

$$\frac{d\rho}{ds}-\frac{1}{2}\rho=e^{s/\lambda}\left(\frac{df}{ds}+\frac{1}{2}f\right).$$

Then (2.4) gives us by integration that

$$\rho(s)=c_0(s)\exp\left[s\left(\frac{1}{\lambda}-\frac{1}{2}\right)\right]-\frac{1-\lambda}{\lambda}e^{\frac{s}{2}}\int_0^s c_0(z)\,e^{\left(\frac{1}{\lambda}-1\right)z}dz.$$

The salt distribution resulting from the gravitational flow is defined by

$$c_1(\xi)=c(\xi,\xi)=\frac{e^{-\gamma\xi}}{\sqrt{\pi\lambda\xi}}\int_0^\infty e^{-\frac{s^2}{4\lambda\xi}}\left[c_0(s)\,e^{\varkappa s}-\varkappa e^{-\varkappa s}\int_0^s c_0(z)\,e^{2\varkappa z}dz\right]ds$$

$$(2\varkappa=1/\lambda-1;\ \gamma=(1-\lambda)^2/(4\lambda)).$$

Examples. 1. For $c_0(\xi) = \exp(-a\xi)$ we get

$$c_1(\xi)=\frac{\varkappa-a}{2\varkappa-a}\exp[a\lambda(a-2\varkappa)\xi]\{1+\mathrm{erf}[(\varkappa-a)\sqrt{\lambda\xi}]\}+\frac{\varkappa}{2\varkappa-a}[1-\mathrm{erf}(\varkappa\sqrt{\lambda\xi})];$$

2. For $c_0(\xi)=\begin{cases}a, & 0<\xi<\xi_0;\\ b, & \xi_0<\xi<\infty\end{cases}$

$$c_1(\xi)=\frac{a}{2}\left[\mathrm{erf}\left(\frac{\xi_0}{2\sqrt{\lambda\xi}}-\varkappa\sqrt{\lambda\xi}\right)+\mathrm{erf}\left(\frac{\xi_0}{2\sqrt{\lambda\xi}}+\varkappa\sqrt{\lambda\xi}\right)\right]+$$

$$+\frac{b}{2}\left[1-\mathrm{erf}\left(\frac{\xi_0}{2\sqrt{\lambda\xi}}-\varkappa\sqrt{\lambda\xi}\right)\right]-\frac{1}{2}\left[(a-b)\,e^{2\varkappa\xi_0}-a\right]\left[1-\mathrm{erf}\left(\frac{\xi_0}{2\sqrt{\lambda\xi}}+\varkappa\sqrt{\lambda\xi}\right)\right].$$

LITERATURE CITED

1. A. A. Rode, "The minimum water content," Pochvovedenie, No. 12 (1965).
2. S. F. Aver'yanov. "Soil permeability in relation to air content," Dokl. Akad. Nauk SSSR, 69, No. 2 (1949).
3. V. I. Pen'kovskii. "Simulation of soil desalination," Zh. Prikl. Mekh. Tekh. Fiz., No. 5, 186-191 (1975).